# A Comprehensive Theory of Yielding and Failure for Isotropic Materials

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Abstract

   A theory of yielding and failure for homogeneous and isotropic materials is given. The theory is calibrated by two independent, measurable properties and from those it predicts possible failure for any given state of stress.  It also differentiates between ductile yielding and brittle failure.  The explicit ductile-brittle criterion depends not only upon the material specification through the two properties, but also and equally importantly depends upon the type of imposed stress state.  The Mises criterion is a special (limiting) case of the present theory.  A close examination of this case shows that the Mises material idealization does not necessarily imply ductile behavior under all conditions, only under most conditions.  When the first invariant of the yield/failure stress state is sufficiently large relative to the distortional part, brittle failure will be expected to occur.  For general material types, it is shown that it is possible to have a state of spreading plastic flow, but as the elastic-plastic boundary advances, the conditions for yielding on it can change over to conditions for brittle failure because of the evolving stress state.  The general theory is of a three dimensional form and it applies to full density materials for which the yield/failure strength in uniaxial tension is less than or at most equal to the magnitude of that in uniaxial compression.

Introduction and Objective

The failure of materials generates research at all length scales from the electronic state to the atomic scale to nano to micro and on to macro (macroscopic) scales. The resulting information is considerably enhanced when the effects at the various scales can be interrelated. Usually the intention is to produce a particular result at the macro scale with this behavior being controlled by the mechanisms operative at the smaller scales. However, it can be difficult to confidently and securely approach the intended macro scale when its characterization is so shrouded in doubt and uncertainty. It would be highly advantageous to have a complete and comprehensive account of failure at the macro scale, one which transcends the various materials classes. The alternative is to have descriptors that are unique to each class or sub-class of materials but with great uncertainty as to the range and limits of validity of each characterization. This latter condition represents the current status.

The objective here is to present and then probe a reasonably complete macroscopic theory of yielding and failure for homogeneous and isotropic materials. Much of the formalism will be synthesized here from various publications that have recently appeared but of necessity have been given in somewhat fragmented and unrelated forms. Using this new formulation, new results will be found for the yielding, plastic flow and failure in several important problems or classes of problems. This communication completes the cycle of recent papers on the failure of materials mentioned above and this also will be the final published paper by the author. In whatever direction future work in the field goes, the present work may help to stimulate further interest and related activity.

It is an evident irony of the history of mechanics that the many books written on the strength of materials basically had almost nothing to do with that subject. Such books were virtually confined to the linear range of elastic behavior with minimal or no attention to failure. This occurred because the understanding of failure as an organized discipline was non-existent. This state continued until the advent of fracture mechanics, about which more will be said later. The sparse historical scene of successful research upon materials failure did have one major prominence and this account should begin by acknowledging the subtle but profound contribution of Coulomb (1773). Mohr (1914) put Coulomb's failure result into a form allowing easy utility. The fact that the Coulomb-Mohr failure form does not successfully account for many of the physical effects does not detract from the efforts of either scientist. In the time frames of their separate works, their grasp of the problem was completely beyond compare. Later, the Mises criterion was given, Mises (1913), but only as an adjunct to a special case of the Coulomb criterion, namely the Tresca form. Both the Mises and Tresca criteria apply only to the yielding of very ductile metals. The Coulomb-Mohr form was intended to apply across the spectrum of materials types, as is the interest here. A history of strength and failure treatments has been given by Paul (1968), and a brief historical summary by Christensen (2004).

At the most elementary level it is sometimes said that ceramics are brittle, many but not all metals are ductile, and some types of polymers are ductile and others are brittle.

While there is a degree of truth in this assertion, it can be extremely misleading, or even worse. An example will be given later wherein a material commonly considered as being completely ductile when placed in a particularly important special state of stress fails in a brittle manner. Relative to failure, all materials can behave either in a ductile or a brittle manner depending upon the state of stress that they are under and other environmental influences.

The two terms, yielding and failure, have imprecise definitions that usually allow a wide latitude of interpretation. This imprecision underlies an uncertain basis of operation. The obvious exception to this situation was the development of fracture mechanics. Fracture mechanics was one of the technical achievements of modern mechanics. A typical application of fracture mechanics involves determining the stability, under imposed stress, of imperfections and stress risers such as cracks and edge notches, holes, attachments etc. In contrast, the means of applying fracture mechanics to the problem of the failure of homogeneous materials under uniform stress states has been far less clear. There are many opinions on this subject, but little substance beyond general statements. Thus, for the failure of homogeneous materials, the integration of fracture mechanics into a more general formalism has not been successfully accomplished in the past.

Failure criteria have usually been formulated in terms of stresses, but over the historical time span, failure criteria have occasionally been postulated in terms of strains. The view here is that trying to specify failure in terms of strains is inappropriate and internally inconsistent. Stress must be used in order to have compatibility with fracture mechanics in the brittle range and with dislocation mechanics in the ductile range. To not have union with these two anchor points of physical reality would be extremely serious. Furthermore, force (stress) is the greatly preferred form for molecular dynamics simulations. Stress, not strain, will be used here for these well grounded reasons.

In the modern era there have been many attempts to find criteria more general than just that for the perfectly ductile response or alternatively the fracture controlled response. The references cited above give many previous references to such works. A sampling of these efforts should include the following. Drucker and Prager (1952) gave a two-parameter yield criterion of conical form in principle stress space. Paul (1968) proposed a three-parameter pyramidal type yield surface. Wronski and Pick (1977) applied Paul's criteria to polymers. Raghava, Caddell and Yeh (1973) proposed a criterion similar to parts of the present forms, and applied it to polymers. Stassi (1967) also discussed similar forms, but without application. Pae (1977) applied a three-parameter criterion to polymers. Wilson (2002) applied the Drucker-Prager criterion to metals. Jaeger and Cook (1979) discussed many three or more parameter models for application to geological materials. None of these approaches possess the combined attributes of involving only a few adjustable parameters, preferably only two, along with the power and flexibility to re-create many different physical effects for many different types of materials. Also, none of these works deal with the essential problem of providing indicators (for any stress state of interest) that differentiate a materials

capability for undergoing ductile flow as opposed to the undesirable outcome of brittle failure.

The present work will give specific meanings to the terms yielding and failure, ones that will offer a useful distinction between them. There are two technical keys to the following developments. These are: (i) the explicit integration of a fracture mechanism into the yielding versus failure formalism, and (ii) the derivation of an explicit criterion that determines whether a failure mode in any particular state of stress is expected to be of ductile or brittle nature. Not surprisingly, these developments (i) and (ii) are found to be interrelated, but still take separate forms in the final set of equations and conditions.

Next the governing relations of this new theory will be given.

<u>Conditions for Yielding and Failure</u>

The references from which various aspects at this new theory of yielding and failure are collected are Christensen (1997, 2000, 2004, 2005, 2006a, 2006b). The initial work in 1997 identified a non-dimensional properties grouping that spanned the range from completely ductile to very brittle behaviors. The year 2000 work was a view of ductile versus brittle behavior, approached in a very mathematical formalism. A broader treatment in 2004 brought in an explicit fracture mechanism and involved comparison with experimental results for a wide variety of materials types. In 2005 an explicit criterion was derived for distinguishing expected brittle failure behaviors from those of plastic yielding, expressed in terms of the imposed stress state and a particular material characteristic. The first 2006 work was an examination of plastic flow potentials, needed when the condition of ductile behavior exists. The second 2006 work involved a detailed comparison with the Coulomb-Mohr and Drucker-Prager (1952) theories. The collection of these individual works and particularly the present amalgam of all of them serve to form this comprehensive account of yielding and brittle failure.

The mathematical conditions to be given in this section for yielding and failure are primarily taken from the above references, but for the background details, the particular references should be consulted. A uniform notation and terminology will be adopted. In this section and the following sections some previously open issues will be clarified and closed, some important special problems will be examined and some critical interpretations given.

The governing yield/failure function for isotropic materials was found by taking a polynomial expansion, through terms of $2^{nd}$ degree, of the invariants of the stress tensor. This procedure gives

$$\alpha \hat{\sigma}_{ii} + \frac{3}{2}(1+\alpha)\hat{s}_{ij}\hat{s}_{ij} \quad \leq \quad 1 \tag{1}$$

where $s_{ij}$ is the deviatoric stress tensor and $\hat{\sigma}_{ij}$ is non-dimensional stress with

$$\hat{\sigma}_{ij} = \frac{\sigma_{ij}}{\kappa}$$

where

$$\alpha = \frac{C}{T} - 1 \quad , \quad 0 \leq \alpha < \infty$$

$$\kappa = C \tag{2}$$

C and T are the uniaxial yield/failure stresses in compression and tension. Alpha, $\alpha$, is the non-dimensional properties grouping that affords special advantages.

When the yield/failure in uniaxial compression is not to be used, $\alpha$ and $\kappa$ will be shown later to take other forms. Relation (1) as written out in terms of components is given in the last section.

Relation (1) is part of the governing criteria, the other essential part is an explicit fracture criterion given by

$$\overset{\wedge}{\sigma}_1 \leq \frac{1}{1+\alpha} \quad \text{if} \quad \alpha \geq 1 \tag{3}$$

or in dimensional form

$$\sigma_1 \leq T \quad \text{if} \quad T \leq \frac{C}{2} \tag{3a}$$

where $\sigma_1$ is the largest principle stress. The relationships of (1) and (3) to some historical criteria as well as their physical interpretations will be given at the end of this section and in the following section.

The two criteria (1) and (3) in the range of $\alpha \geq 1$ are both employed to find whichever of the two is the more limiting. Also, the two relations (1) and (3) are in some sense coupled through the appearance of $\alpha$ in both of them. For $\alpha \leq 1$ relation (1) stands alone. It is best to view the variation of $\alpha$ on a log scale having the range from $-\infty$ to $\infty$. Then the value $\alpha = 1$ in (3) designates the center of the log scale range, $\log \alpha = 0$. This center point is where the fracture criterion (3) begins to take effect. Compared with the yield/failure criterion (1), the fracture criterion (3) can be shown to only have an infinitesimal difference from (1) in the immediate vicinity of $\alpha = 1$, $\log \alpha = 0$, but it becomes of increasing significance as $\alpha$ increases beyond this value.

The fracture restriction (3) naturally corresponds to brittle behavior, but under certain conditions relation (1) also can embody brittle failure characteristics. This is why the form in (1) is termed as the yield/failure function. This condition is shown schematically in Fig. 1 for biaxial stress states with $\alpha = 2$. The intersection of the yield function in (1) and the fracture condition in (3) establishes the division into ductile and brittle regions. It is seen in Fig. 1 that the mean normal stress part of the failure stress state locates the position of the ductile/brittle dividing line. This ductile-brittle delineation generalizes directly to tri-axial stress conditions and gives the D-B criterion as

$$\overset{\wedge}{\sigma}_{ii} < \frac{2-\alpha}{1+\alpha} \quad , \quad Ductile$$

$$\overset{\wedge}{\sigma}_{ii} > \frac{2-\alpha}{1+\alpha} \quad , \quad Brittle \tag{4}$$

where $\overset{\wedge}{\sigma}_{ii}$ is found from the stresses satisfying relation (1) as an equality and $\alpha$ is taken over its full range. That is, $\overset{\wedge}{\sigma}_{ii}$ is the first stress invariant (mean normal stress) of the yield/failure stress state. The right hand side of (4) specifies the material type through the value of $\alpha$, while the left hand side specifies the part of the yield/failure stress state that controls whether the behavior will be brittle or ductile.

The region near the dividing line between ductile and brittle behavior specified by (4) is in reality likely to be transitional rather than discontinuous. Furthermore, this region is more likely to be dominated by the brittle inclination than by the plastic flow tendency. As a separate matter, any time the fracture criterion (3) is controlling, that by definition is a brittle behavior. These general behaviors are similar to those modeled by Harlin and Willis (1988) wherein they used macroscopic criteria with the involvement of tri-axial stress conditions to distinguish the ductile from the brittle response of growth activated cracks.

Relations (1)-(4) comprise the entire yield/failure criteria. Only two properties are involved, the uniaxial tensile and compressive yield/failure values. Strain hardening could be involved in this methodology, but it is not included here in order to focus upon the essential but idealized aspects of the ductile versus brittle behavior. The yield stress in a ductile behavior is taken to be that at the point of major deviation between the previous elastic region and the following plastic flow region. The idealized behavior is as shown in Fig. 2. It is clear that in the context of the brittle and ductile behaviors of Fig. 2 stress alone cannot distinguish between the three conditions of: (i) brittle failure, (ii) ductile yield or (iii) ductile failure. Thus, it is necessary to have the separately determined criterion to distinguish brittle from ductile response. The complete theory does not predict the strain at failure after ductile flow, only that such a flow state exists. In some cases the amount of plastic flow before failure may be very small and it may be somewhat difficult to distinguish these cases from brittle failure.

It is to be expected that the ductile-brittle criterion (4) would be controlled by the mean normal stress part of the total stress tensor at yield or failure. This is similar to and in fact related to the effect of temperature on the same ductile, brittle behaviors, see also Harlin and Willis (1988). Temperature and pressure are the two most fundamental control variables for such effects. Again it is emphasized that in physical reality there would not be an abrupt dividing line between the two states, but rather a region or band of most rapid variation from ductile to brittle.

To complete this theory it is necessary to specify the governing characteristics of plastic flow when criteria (4) specifies that the ductile regime controls. Following Hill (1950) the plastic flow potential is necessarily taken to be different from the yield/failure function, in (1), thereby leading to a non-associative form. The decomposition of strain components into elastic and plastic parts then has the plastic strain increments given by

$$\overset{\cdot}{\varepsilon}_{ij}^{p} = \lambda \frac{\partial G}{\partial \sigma_{ij}}$$

(5)

where G is the independent plastic flow potential. It was found (in the previous references) that for homogeneous materials G takes the form

$$G = \frac{\alpha A}{1 + \alpha} \hat{\sigma}_{kk} + \frac{3}{2} \hat{s}_{ij} \hat{s}_{ij}$$
(6)

where A is a constant, an order of magnitude less than one, and with considerable support for a value of about $A \cong 1/15$, from the data of Richmond and colleagues, Spitzig, Sober and Richmond (1975), Spitzig, Sober and Richmond (1976) and Spitzig and Richmond (1979). The hypothetical value, A = 1, would correspond to the associative form. The dilatational term in (6) is due to the mechanism by which materials in plastic flow create a small amount of voids and vacancies.

In most (but not all) situations and for most homogeneous and isotropic materials the first term in (6) is small compared with the second term because of the smallness of constant A and can be neglected leaving

$$G \cong \hat{s}_{ij} \hat{s}_{ij}$$
(7)

which gives the plastic flow potential as being completely distortional in character. In (7) the 3/2 factor in (6) is absorbed into $\lambda$ in (5).

This is a two-property theory calibrated by the uniaxial tensile and compressive yield/failure values. It should be noted that the uniaxial compressive test can be difficult to perform. See Lassila et al (2002) for a detailed account of the measures and care needed to produce a reliable test for uniaxial compression. An example of an alternative set of two independent yield/failure prescriptions is that of uniaxial tension and uniaxial tension under pressure. In this case it can be shown that $\alpha$ and $\kappa$ are given by

$$\alpha = \frac{2\Theta}{3 - \Theta} \quad , \quad \alpha \leq 1$$
(8)

where

$$\Theta = \left. \frac{dT}{dp} \right|_{p=0}$$

and then

$$\kappa = (1 + \alpha)T \big|_{p=0}$$

with p being the pressure and T being the uniaxial tensile yield/failure stress superimposed upon p. The restriction in (8) to the range $\alpha \leq 1$ can be removed, but with some additional involvement of fracture relation (3) in the range $\alpha \geq 1$.

Finally, it should be noted that if the fracture criterion (3) were used as a stand-alone criterion it would correspond to the maximum normal stress criterion favored by Rankine and by Lamé. That criterion did not succeed as a single, all encompassing criterion, but it certainly has a special and vital role to play here as a restriction tightly coordinated with the yield/failure function (1). With regard to macroscopically homogeneous materials, the fracture criterion (3) is here seen as effectively a Mode I fracture event activated by whatever inhomogeneities control the behavior on the micro-scale. Since this theory is only for macroscopically homogeneous and isotropic materials, the microscale inhomogeneities possibly causing fracture must not disturb the macroscale conditions of homogeneity and isotropy. It also follows that any initial microscale porosity must be small.

General Characteristics

Cases represented by the limiting values of $\alpha$ will be discussed as well as some intermediate values of $\alpha$. The case of $\alpha = 0$ leaves the yield/failure function in (1) as reducing to the Mises criterion. This $\alpha = 0$ case will be referred to as that of a Mises material. It is commonly accepted that a Mises material is the example of a perfectly ductile solid, this implying that there exists ductility under all conditions. However, the present results show that this notion is not correct. The ductile-brittle criterion (4) is not always satisfied as being ductile for $\alpha = 0$. In fact, it can be shown that criterion (1) and (4) reveal that in principle stress space the infinite cylindrical form of the Mises criterion has ductile and brittle regions as shown in Fig. 3. The case of uniaxial tension is on the ductile side of the D-B dividing plane in Fig. 3 as are most of the common stress states such as unequal biaxial stresses. But the special case of equal biaxial tension is right on the dividing line between the ductile and brittle regions. More will be said about this important case later. As seen from Fig. 3 any stress state at yield/failure with a sufficiently large tensile hydrostatic component will be brittle. An example is as follows. When a high intensity compressive dilatational wave reflects from a free surface it creates a large tri-axial tensile stress state condition. This can cause micro-scale void nucleation as a precursor to the macro-scale brittle failure condition of spallation, even for what are normally thought of as ductile materials.

For values of $\alpha > 0$ the yield/failure function (1) is a paraboloid in principle stress space, Fig. 4. Its axis makes equal angles with the three principle stress axes. For $\alpha > 1$ the fracture criterion (3) comes into effect. Right at $\alpha = 1$ the three planes prescribed by the fracture criterion (3) are just tangent to the yield/failure paraboloid. As $\alpha$ increases beyond $\alpha = 1$ the fracture criterion takes three slices out of the paraboloid. These are here called fracture cut offs. An illustration of these fracture cut-offs at $\alpha = 2$ is shown in Fig. 1 for biaxial stress, with this value of $\alpha$ being typical of cast iron. The paraboloid is divided into ductile and brittle regions by its intersection with the plane normal to its axis prescribed by (4). This ductile-brittle division must also deviate around the fracture cut-offs prescribed by (3).

The limiting case of $\alpha \to \infty$ is the brittle limit with the tensile strength T being negligible compared with the compressive strength C. This limiting case is perhaps best viewed through the shear stress at yield/failure. From (1) the shear stress $\tau$ at $\alpha \to \infty$ is very small compared with C. Now consider a state of shear stress $\tau$ superimposed upon pressure p. From (1) at $\alpha \to \infty$ the yield/failure shear stress under pressure is given by

$$\tau = \sqrt{pC} \tag{9}$$

The associated ductile-brittle criterion (4) gives the result

$$p > \frac{C}{3} \quad , \quad \text{Ductile}$$

$$\tag{10}$$

$$p < \frac{C}{3} \quad , \quad \text{Brittle}$$

The fracture criterion (3) gives

$$\tau = p \tag{11}$$

which is by definition brittle.

Conditions (9)-(11) give the complete behavior as

$$\tau = p \text{ for p} \leq C \text{ , Brittle}$$

and
$$\tag{12}$$

$$\tau = \sqrt{pC} \text{ for p} \geq C \text{ , Ductile}$$

Relations (12) are shown in Fig. 5. Again, at p = 0 the material cannot sustain any shear stress. Sufficiently large pressure supports a ductile behavior in shear. The case just considered corresponds to the condition $\sigma_{ii} \leq 0$. For the condition of positive mean normal stress, $\sigma_{ii} > 0$, as $\alpha \to \infty$, relation (1) can be used to show that it becomes just

$$\sigma_{ii} \leq T \tag{13}$$

This relation is more restrictive than the fracture condition (3) and from (4) it is of a brittle nature. The physical interpretation of (13) is that approaching this brittle limit of $\alpha \to \infty$ any stress state with positive $\sigma_{ii}$ satisfying (13) as an equality causes the material to disassociate and disintegrate. Although it is important to verify reasonable and rational behavior in the limiting case of $\alpha \to \infty$, probably the practical maximum range for applicability of this theory would be for ceramics and glasses. These materials would have values of $\alpha$ from about 5 to 25.

Now, two general examples will be given, first an equal biaxial stress state and then unequal biaxial stresses in a 2:1 ratio. For equal biaxial stresses let

$$\sigma_{11} = \sigma_{22} = \sigma \tag{14}$$

The yield/failure criterion (1) gives

$$\hat{\sigma} = \frac{1}{(1+\alpha)} \left[ -\alpha \pm \sqrt{1 + \alpha + \alpha^2} \right] \tag{15}$$

For the case of tension and at $\alpha = 0$ then (15) gives $\hat{\sigma} = 1$. With this result then it follows that the left hand side of (4) is given by

$$\hat{\sigma}_{ii} = 2$$

The right hand side of the ductile-brittle criterion (4) at $\alpha = 0$ is simply 2. Thus at $\alpha = 0$ and in equal biaxial tension the yield/failure state is borderline between being ductile or brittle. For $\alpha > 0$ it is always brittle thus for all values of $\alpha$

Equal Biaxial Tension $\Rightarrow$ Brittle

Using (15) in the case of compression with criterion (4) then gives for all values of $\alpha$

Equal Biaxial Compression $\Rightarrow$ Ductile

The significance of this result that equal biaxial tension is brittle is that this is the stress state existing in thin spherical pressure vessels. The failure is of brittle type, even though the material, such as aluminum for example, is nominally considered as being ductile. Thus thin spherical pressure vessels can represent a considerable safety hazard no matter how ductile the composing material may be, in accordance with practical experience.

The example just considered suggests looking at the case of cylindrical pressure vessels. In the thin cylindrical portion of the pressure vessel the stress state is given by biaxial stresses with

$$\sigma_{11} = \sigma$$

$$\sigma_{22} = \frac{\sigma}{2}$$

(16)

The yield failure criterion (1) gives

$$\hat{\sigma} = \frac{1}{(1+\alpha)}\left[-\alpha \pm \sqrt{\alpha^2 + \frac{4}{3}\alpha + \frac{4}{3}}\right]$$

(17)

Using the stress states for tension and compression from (17) in the ductile-brittle criterion (4) gives

Tension, $\alpha < \frac{1}{2}$ $\Rightarrow$ Ductile

Tension, $\alpha > \frac{1}{2}$ $\Rightarrow$ Brittle

and

Compression $\Rightarrow$ Ductile

for all values of $\alpha$. In the tensile case, the D-B dividing line at $\alpha = 1/2$ corresponds to T/C = 2/3, which is right in the range of many polymers.

In both of these latter two examples symbolic of thin shell pressure vessels, the fracture criterion (3) is not as limiting as the results shown above from the yield/failure

criterion (1).  These examples are for the two different bi-axial stress states.  When interpreted for application to thin shell pressure vessels, only the tensile conditions would be meaningful.  Instability would supercede the compressive yield results.

Lame´ Problems

Now consider more complex three-dimensional problems with non-uniform stresses. The Lame´ problems for thick cylinders and spheroids are the classical problems of this type. In particular a thick spherical annulus with internal or external pressure will be considered.

The internal pressurization of the spherical annulus is as shown in Fig. 6. In the case of an ideal Mises material the complete solution is given by Hill (1950). The solution for the more general material model involving yield and or failure governed by (1)-(7) will be developed here. For sufficiently small pressures the entire region will be elastic, but at some pressure the material at the inner radius will commence yielding, and as pressure further increases the elastic plastic boundary at $r = \rho$, Fig. 6, will move outward.

The stress solution for the entire elastic deformation case is given by

$$\sigma_r = -p \left[ \frac{\left(\frac{b}{r}\right)^3 - 1}{\left(\frac{b}{a}\right)^3 - 1} \right]$$

and                                                                                     (18)

$$\sigma_\theta = \sigma_\phi = p \left[ \frac{\frac{1}{2}\left(\frac{b}{r}\right)^3 + 1}{\left(\frac{b}{a}\right)^3 - 1} \right]$$

where p is the internal pressure.

Now find the pressure at which yielding first commences at $r = a$. The yield/failure function (1) becomes

$$\alpha\left(\hat{\sigma}_r + 2\hat{\sigma}_\theta\right) + (1+\alpha)\left(\hat{\sigma}_r - \hat{\sigma}_\theta\right)^2 = 1 \qquad (19)$$

Combining (18) and (19) at $r = a$ gives the pressure for initial yielding as

$$\hat{p} = \frac{2(1-\lambda)}{3(1+\alpha)}\left[-\lambda\alpha + \sqrt{1+\alpha+\lambda^2\alpha^2}\right] \qquad (20)$$

where

$$\lambda = \left(\frac{a}{b}\right)^3$$

and

$$\hat{p} = \frac{p}{\kappa}$$

The symbol $\lambda$ used here should not be confused with that in (5).

Next, the question of whether this initial yielding is actually of ductile yielding or brittle failure must be answered. Using the stresses at r = a in the ductile-brittle criterion (4) gives

$$\frac{3\hat{p}}{\left(\dfrac{b}{a}\right)^3 - 1} < \frac{2 - \alpha}{1 + \alpha} \quad , \quad \text{Ductile}$$

$$(21)$$

$$\frac{3\hat{p}}{\left(\dfrac{b}{a}\right)^3 - 1} > \frac{2 - \alpha}{1 + \alpha} \quad , \quad \text{Brittle}$$

where $\hat{p}$ is given by (20). Interest here will be confined to cases of $\alpha \leq 1$, thus the fracture criterion (3) does not enter the considerations. Materials with $\alpha > 1$ would almost never be used in applications of internal pressurization since they would be too brittle. Combining (20) and (21) gives

$$\left(\frac{a}{b}\right)^3 < \frac{1 - \dfrac{\alpha}{2}}{\sqrt{1 - \alpha + \alpha^2}} \quad , \quad \text{Ductile}$$

$$(22)$$

$$\left(\frac{a}{b}\right)^3 > \frac{1 - \dfrac{\alpha}{2}}{\sqrt{1 - \alpha + \alpha^2}} \quad , \quad \text{Brittle}$$

For example for $\alpha = 1$ (22) became

$$\frac{a}{b} < 0.794 \quad , \quad \text{Ductile}$$

$$(23)$$

$$\frac{a}{b} > 0.794 \quad , \quad \text{Brittle}$$

The ductile case will be considered here. If cases with $\alpha \geq 2$ were to be considered they would be found to always be of brittle failure type. For values of $\alpha$ between 1 and 2 the

previous criterion can be used to find the values of a/b for which the ductile or brittle condition applies.

For pressures greater than that in (20) the medium deforms into a plastic interior region and an elastic exterior region, with the division at $r = \rho$. For the elastic region the stresses are given by

$$\sigma_r = -A\left[\left(\frac{b}{r}\right)^3 - 1\right]$$

$$\sigma_\theta = \sigma_\phi = A\left[\frac{1}{2}\left(\frac{b}{r}\right)^3 + 1\right]$$

(24)

where A is a constant to be determined. At $r = \rho$ the elastic stresses must satisfy the yield/failure criterion (19), which determines A as

$$\frac{A}{\kappa} = \frac{2\left(\frac{\rho}{b}\right)^3}{3(1+\alpha)}\left[-\alpha\left(\frac{\rho}{b}\right)^3 + \sqrt{1+\alpha+\left(\frac{\rho}{b}\right)^6\alpha^2}\right]$$

(25)

For later use it is necessary to have $\hat{\sigma}_r$ at $r = \rho$ which is found as

$$\sigma_r\Big|_{r=\rho} = \frac{2\left[1-\left(\frac{\rho}{b}\right)^3\right]}{3(1+\alpha)}\left[\alpha\left(\frac{\rho}{b}\right)^3 - \sqrt{1+\alpha+\left(\frac{\rho}{b}\right)^6\alpha^2}\right]$$

(26)

With the elastic region determined, now the stresses in the plastic region will be found, being careful to only do this under conditions of ductile behavior. The governing equilibrium equation is

$$\frac{d\sigma_r}{dr} + \frac{2(\sigma_r - \sigma_\theta)}{r} = 0$$

(27)

Use the yield function (19) to solve for $\hat{\sigma}_\theta$ and substitute that into (27) giving

$$(1+\alpha)\frac{d\hat{\sigma}_r}{dr} + \frac{2}{r}(\alpha - \eta) = 0$$

(28)

where

$$\eta = \sqrt{1+\alpha+\alpha^2 - 3\alpha(1+\alpha)\hat{\sigma}_r}$$

(29)

This nonlinear differential equation can be integrated to give $\hat{\sigma}_r$ as a function of r. Carrying out this process, and evaluating the constant of integration such that at $r = \rho$, $\hat{\sigma}_r$ from (28) and (29) is continuous with the elastic region stress (26) then gives

$$3\alpha Ln\frac{r}{\rho} = -\left[\eta + \alpha Ln(-\alpha + \eta)\right]$$
$$+\left[\eta_1 + \alpha Ln(-\alpha + \eta_1)\right] \tag{30}$$

where

$$\eta_1 = \eta\Big|_{\hat{\sigma}_r = \hat{\sigma}_r|_{r=\rho}} \tag{31}$$

Symbol $\hat{\sigma}_r\big|_{r=\rho}$ in (31) is given by (26).

The boundary conditions at $r = a$ is

$$\hat{\sigma}_r = -\hat{p} \quad \text{at} \quad r = a \tag{32}$$

which when substituted into (30) gives

$$\left[\eta_2 + \alpha Ln(-\alpha + \eta_2)\right] = 3\alpha Ln\frac{\rho}{a} + \left[\eta_1 + \alpha Ln(-\alpha + \eta_1)\right] \tag{33}$$

where

$$\eta_2 = \eta\Big|_{\hat{\sigma}_r = -\hat{p}} \tag{34}$$

Now the full stress solution in the plastic region has been found, subject to the condition that ductile plastic flow occurs, rather than brittle failure. This will be illustrated in a particular example. Equation (33) provides the relationship between the applied internal pressure $\hat{p}$ and the location of the elastic-plastic boundary, $\rho$. Once this is known then the stress $\hat{\sigma}_r$ follows from (29) and (30) as a function of r. The other stresses follow from (24) and (25).

Take a particular example of $a/b = 1/3$ and $\alpha = 1$ which is typical of some metals and polymers. The ductile-brittle criterion (22) applied to the elastic region at $r = \rho$ gives

$$\left(\frac{\rho}{b}\right)^3 < \frac{1}{2} \quad , \quad \text{Ductile}$$

(35)

or

$$\left(\frac{\rho}{b}\right) < 0.794 \quad , \quad \text{Ductile}$$

Thus plastic flow ceases when sufficient pressure has caused the elastic plastic boundary to reach the value in (35). Brittle failure follows thereafter.

For this particular example the solution given here determines that for first yield

$$\hat{p} = 0.442 \quad \text{First Yield}$$

and the pressure at which $\rho$ becomes that shown in (35) is

$$\hat{p} = 1.70 \quad \text{Termination of Plastic Flow}$$
$$\text{Inception of Brittle Failure}$$

For comparison the solution for a Mises material, $\alpha = 0$, gives

$$\hat{p} = 0.642 \quad \text{First Yield}$$

and

$$\hat{p} = 2.20 \quad \text{Fully Plastic Yield}$$

The stress components in this solution example for $\alpha = 1$ follow directly from the previous results. The plastic strain increments can similarly be found by using the plastic flow potential (6) or the simplified form (7).

An important characteristic has emerged from this solution and example. It is that in elastic, plastic flow problems, as the flow progresses and the boundary between the two regions changes, so to can the conditions existing at the elastic plastic boundary switch over from that of plastic flow to that of brittle failure. Thus the conditions for plastic flow, at a particular configuration, can cease to exist and plastic flow is interrupted by brittle failure. Although this behavior was found from the present idealized theoretical formulation, physical reality can be expected to at least show some features or aspects of this behavior. This possibility of switching the failure mode and type follows from the dependence of the ductile-brittle criterion upon the prevailing state of stress as well as the material type. This complex aspect of physical behavior with regard to the moving elastic plastic boundary appears to have been unrecognized before now.

The corresponding and companion problem of a thick spherical annulus subjected to external pressure can be solved by the same general method as that just discussed. Yielding first occurs on the inner surface at r = a and the external pressure to cause this is given by

$$\hat{p} = \frac{2}{3}\frac{(1-\lambda)}{1+\alpha}\left[\alpha + \sqrt{1+\alpha+\alpha^2}\right] \tag{36}$$

where $\lambda = (a/b)^3$ as before. An examination of the ductile-brittle criterion (4) reveals that the further deformation is always that of ductile plastic flow for all values of $\alpha$, in contrast to the complex situation just discussed for the internal pressurization problem, where plastic flow can be interrupted by brittle failure under certain conditions. An example with a/b = 1/3 and $\alpha = 1$, (36) gives the external pressure at first yielding as $\hat{p} = 0.877$, about twice the value required by the internal pressurization case.

Although the present solutions show that the resulting stresses take more complicated forms than in the simplified Mises material case, it is still completely practical to obtain such results. The formulation is well posed and amenable to numerical solution in complex problems.

Failure Surface Orientations

The orientations of failure surfaces reveal specific patterns and signs of behavior. Certainly brittle material orientations are expected to be much different from ductile material orientations. This topic will now be examined with the objective to determine the failure surface orientations for uniaxial tension and compression as a function of the material type, specified by the value of $\alpha$ in (2). Failure surface orientations in this context have been studied before, Christensen (2005), but there was some uncertainty as to whether the yield/failure function or the plastic flow potential should be used in certain of the operations. This question can now be formally and finally answered.

The usual plastic increment under ductile flow conditions is already stated as (5) where G is the plastic flow potential. The failure that occurs at the end of a ductile flow process is not simply a continuation of the previous plastic flow. It is an inherently unstable process, and it is here taken to be controlled by the yield/failure function, rather than the plastic flow potential. Take the increment of the failure strain, $\overset{\bullet f}{\varepsilon}_{ij}$, to be given

$$\overset{\bullet f}{\varepsilon}_{ij} = \Lambda \frac{\partial f()}{\partial \sigma_{ij}} \tag{37}$$

where f() is the yield/failure function on the left had side of (1). This should not be confused with using the associated flow rule because that terminology only applies to the stable ductile plastic flow. The form (37) is here being applied to the final failure process.

For uniaxial tension, (1) and (37) then give

$$\overset{\bullet f}{\varepsilon}_{11} = \Lambda(2 + \alpha)$$

$$\tag{38}$$

$$\overset{\bullet f}{\varepsilon}_{22} = \overset{\bullet f}{\varepsilon}_{33} = \Lambda(-1 + \alpha)$$

For uniaxial compression (1) and (37) give

$$\overset{\bullet f}{\varepsilon}_{11} = -\Lambda(2 + \alpha)$$

$$\tag{39}$$

$$\overset{\bullet f}{\varepsilon}_{22} = \overset{\bullet f}{\varepsilon}_{33} = \Lambda(1 + 2\alpha)$$

Now take a rotated coordinate system $x_i'$ such that $1'$ is in the plane of the failure surface and $2'$ is normal to it. Let angle $\phi$ be the angle between the axial direction, 1, and $1'$ in

the failure surface. Then the failure strain increments in the rotated coordinates are given by

$$\frac{\left(\dot{\varepsilon}_{11}^{f}\right)'}{\Lambda} = (2+\alpha)\cos\phi - (1-\alpha)\sin^2\phi$$

$$\frac{\left(\dot{\varepsilon}_{22}^{f}\right)'}{\Lambda} = (2+\alpha)\sin^2\phi - (1-\alpha)\cos^2\phi \tag{40}$$

$$\frac{\left(\dot{\varepsilon}_{12}^{f}\right)'}{\Lambda} = 3\sin\phi\cos\phi$$

Take the failure strain increment $\left(\dot{\varepsilon}_{11}^{f}\right)'$ in the plane of the failure surface as being non-active and vanishing while the other two increments in (40) give strains that undergo unlimited change in the failure process. Setting $\left(\dot{\varepsilon}_{11}^{f}\right)' = 0$ then gives the orientation of the failure surface as

$$\tan\phi = \sqrt{\frac{2+\alpha}{1-\alpha}} \quad , \quad \alpha \leq 1, \text{ Tension} \tag{41}$$

At $\alpha = 1$ relation (41) gives $\phi = 90°$. For $\alpha > 1$ the fracture cutoff (2) comes into effect giving brittle failure which results in

$$\phi = 90° \quad , \quad \alpha \geq 1 \quad , \text{ Tension} \tag{42}$$

For the case of uniaxial compression the rotation of coordinates gives

$$\frac{\left(\dot{\varepsilon}_{11}^{f}\right)'}{\Lambda} = -(2+\alpha)\cos^2\phi + (1-2\alpha)\sin^2\phi \tag{43}$$

Similarly to the tensile case, for this failure strain increment in the plane of the failure surface taken as inactive and vanishing while the other 2 strain components grow to unbounded values then gives the orientation of the compressive failure surface as

$$\tan\phi = \sqrt{\frac{2+\alpha}{1+2\alpha}} \quad , \quad Compression \tag{44}$$

The failure surface orientations are shown in Fig. 7 as a function of $\alpha$. At $\alpha = 0$ (a Mises material) the failure surface orientation is at the octahedral angle for both tension and compression.

The uniaxial tensile case shown in Fig. 7 has the ductile versus brittle regions as shown. This behavior coordinates perfectly with the ductile-brittle criterion (4) which shows the change over at $\alpha = 1$ for the case of uniaxial tension.

In the case of uniaxial compression the ductile-brittle criterion (4) is in the ductile range of behavior for all values of $\alpha$ except that it becomes borderline between ductile and brittle as $\alpha \to \infty$. It is interesting to determine the stress components on the failure surface at the orientations given by (44). Let $\sigma$ be the normal stress and $\tau$ be the shear stress on the failure surface. Using (44) and the appropriate coordinate rotations for stress it is found that

$$\hat{\sigma} = \frac{-(2+\alpha)}{3(1+\alpha)}$$

and $\tag{45}$

$$\hat{\tau} = \frac{\sqrt{(1+2\alpha)(2+\alpha)}}{3(1+\alpha)}$$

The two failure surface stresses in (45) for values at $\alpha = 0$, $\alpha = 1$, and $\alpha \to \infty$ are given in Table 1. It must be remembered that $\hat{\sigma} = \sigma/\kappa$ and that $\kappa$ would vary greatly over the range of the $\alpha$'s. These non-dimensional stresses have a special variation with $\alpha$. The shear stress shown in Table 1 hardly varies over the full range of $\alpha$ while the magnitude of the normal stress diminishes with increasing $\alpha$. The latter variation can be interpreted as the indication that failure surface orientation is such that the compressive normal stress on the failure surface tends toward small magnitudes for large values of $\alpha$, but has larger magnitudes of $\hat{\sigma}$ at the smaller values of $\alpha$. In other words, the magnitude of the normal stress is more controlling at large $\alpha$'s and less controlling at small $\alpha$'s, which is intuitively expected. Also shown in Table 1 is the ratio $\tau/|\sigma|$. Note that $\tau/|\sigma|$ has a type of anti-symmetry relative to the central point at $\log \alpha = 0$.

Finally it is perhaps obvious that the yield/failure function rather than the plastic flow potential was the correct quantity to use in finding the failure orientations. If the plastic flow potential (7) were used, the failure surface orientations would have been unvarying with respect to $\alpha$, always at the octahedral angle and the same in compression as in tension.

Conclusions

The previous results will not be summarized here other than to broadly say that this is an approach to the difficult problem of characterizing yielding and failure for general materials. The central focus has been to distinguish brittle failure from ductile yielding type response. It has always been known that this discrimination must depend upon a specification of the material type but the present work also shows that this distinction crucially depends upon the type of stress state under consideration. For example, shear stress gives a different result than does uniaxial tension. When one brings in the full stress tensor, the number of possible combinations becomes boundless and a generalized approach becomes necessary. These resulting special cases display a great variety of different ductile-brittle transition circumstances, many of which have been examined here and shown to be of importance.

It is of some relevance to consider which single stress state, if any, best characterizes a simplified but realistic form of the ductile-brittle characteristics of materials. Uniaxial tension certainly is the logical choice. From the yield/failure criterion (1), for uniaxial tension it follows that

$$\hat{\sigma}_{11}^{T} = \frac{1}{1+\alpha} \tag{46}$$

Substituting this into the ductile-brittle criterion (4) gives

$$\alpha < 1 \quad , \quad \text{Ductile} \tag{47}$$

or

$$\frac{T}{C} > \frac{1}{2} \quad , \quad \text{Ductile} \tag{48}$$

For $\alpha > 1$, the response in this stress state is brittle. This gives the transition in failure type for uniaxial tension as being at the value $\alpha = 1$. This is the same value as for the inception of the fracture criterion in (3). The value of $\alpha = 1$ is at the center of the log $\alpha$ scale. For simple shear stress, criterion (4) directly shows that its transition is at $\alpha = 2$. Other stress states show widely varying values of $\alpha$ at which their ductile to brittle transitions occur.

At an application level, often no attempt is made to distinguish ductile and brittle responses, other than by intuitive guidelines. In this situation failure would be considered in an inclusive sense as encompassing both yielding and complete rupture. Then the previous results could simply be designated as generalized failure criteria. In this case everything needed is directly specified by conditions (1)-(3), rewritten here in terms of components as

$$\left(\frac{1}{T}-\frac{1}{C}\right)\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right)$$

$$+\frac{1}{TC}\left\{\frac{1}{2}\left[\left(\sigma_{11}-\sigma_{22}\right)^2+\left(\sigma_{22}-\sigma_{33}\right)^2+\left(\sigma_{33}-\sigma_{11}\right)^2\right]\right.$$

$$\left.+\ 3\left(\sigma_{12}{}^2+\sigma_{23}{}^2+\sigma_{31}{}^2\right)\right\}\leq 1 \tag{49}$$

and

$$\sigma_1 \leq T \quad \text{if} \quad T \leq \frac{C}{2} \tag{50}$$

where $\sigma_1$ is the largest principle stress. Whichever of (49) or (50) is the more limiting determines the failure condition. Relations (49) and (50) are as easy to use as are the Mises or Tresca criteria, but (49) and (50) have far greater generality. The Mises criterion corresponds to (49) with $T = C$ while (50) is seen to be inapplicable at $T = C$.

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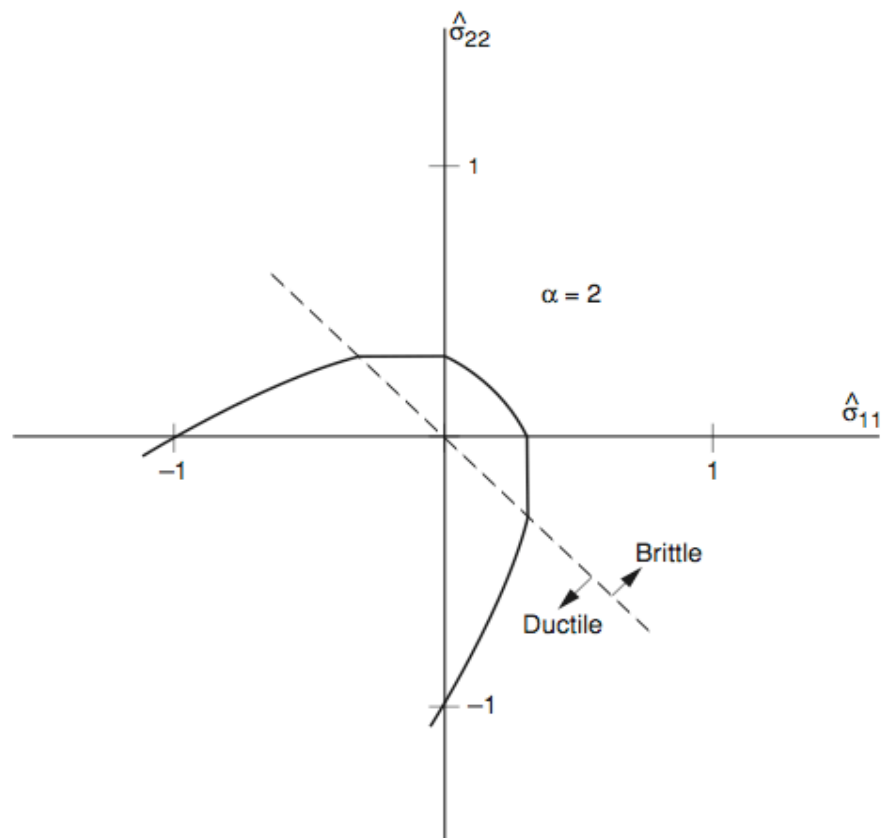| $\alpha$ | 0 | 1 | $\infty$ |
|---|---|---|---|
| $\log \alpha$ | $-\infty$ | 0 | $\infty$ |
| $\hat{\sigma}$ | -2/3 | -1/2 | -1/3 |
| $\hat{\tau}$ | $\sqrt{2}/3$ | ½ | $\sqrt{2}/3$ |
| $\tau/|\sigma|$ | $1/\sqrt{2}$ | 1 | $\sqrt{2}$ |
| $\log(\tau/|\sigma|)$ | -0.151 | 0 | 0.151 |

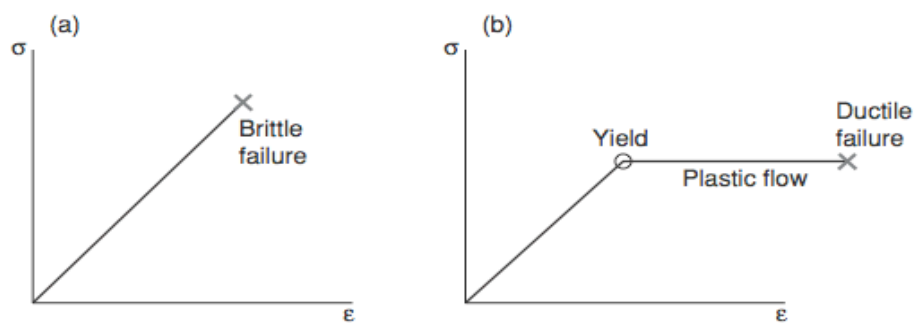Table 1.  Stresses on Failure Surfaces, Eqs. (45)

Figure 1     Biaxial Stress State
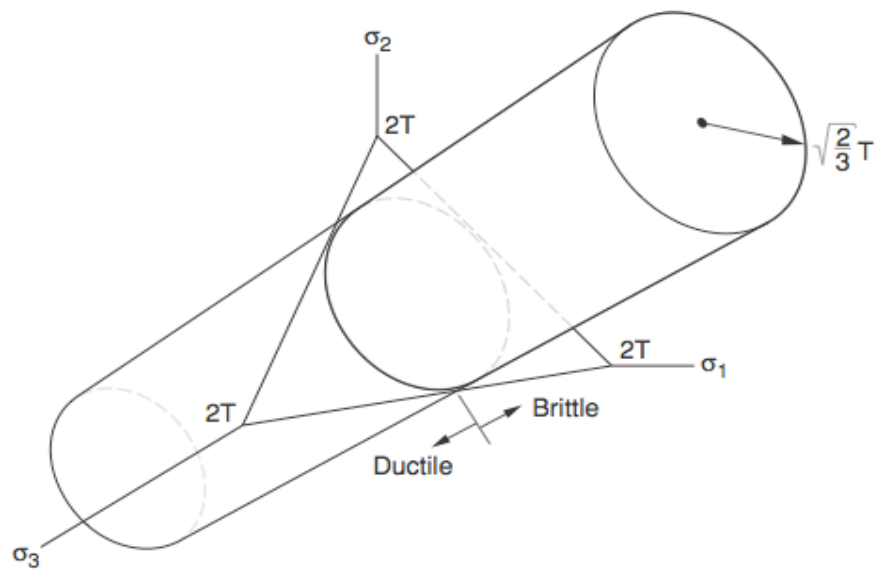
Figure 2    Brittle Failure and Ductile Yield/Failure
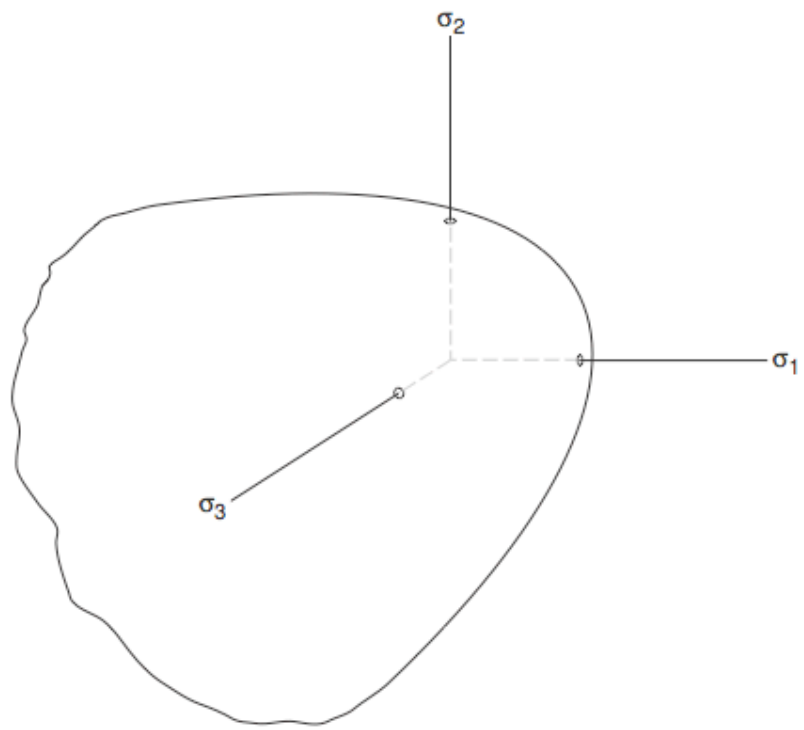
Figure 3    Mises Material, $\alpha = 0$, Ductile Yield
and Brittle Failure Regions
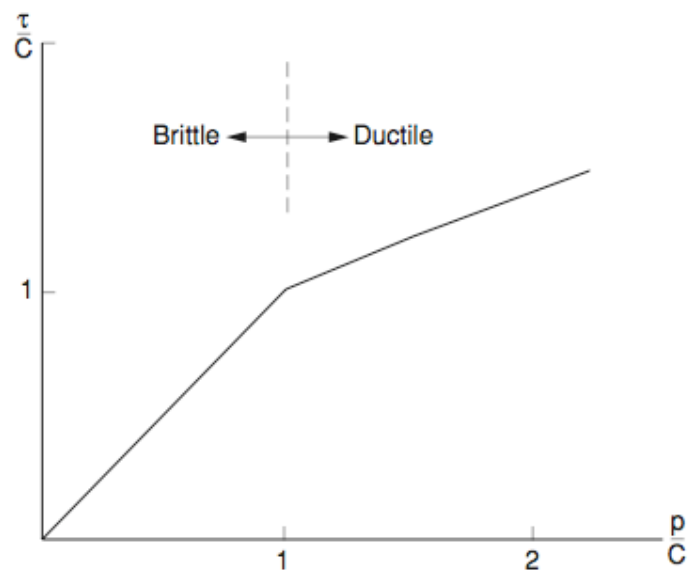
Figure 4    Yield/Failure Paraboloid

Figure 5    Shear Stress Superimposed Upon
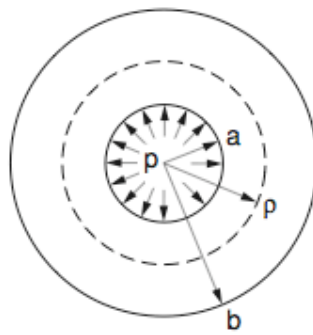Presure, $\alpha \rightarrow \infty$ Limit, Eq. (12)
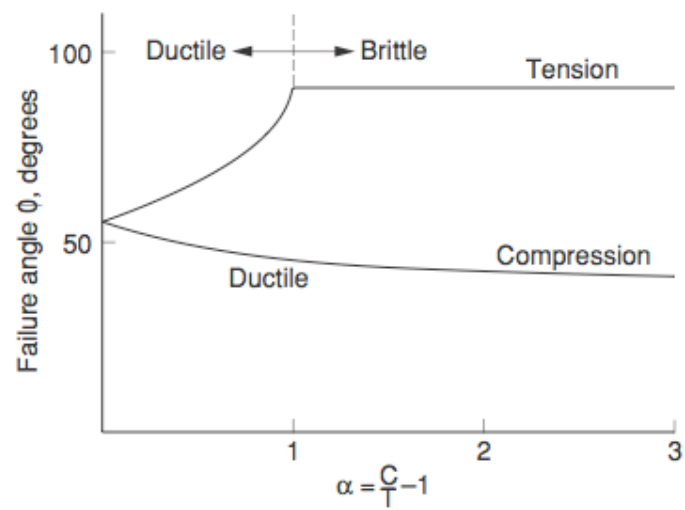
Figure 6    Spherical Annulus with Internal Pressure

Figure 7    Failure Angles, Uniaxial Tension and Compression